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FREQUENCY CHARACTERISTIC METHOD
IN THE THEORY OF REGULATION

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[Figures are appended.]

Introduction

This article gives a basic method for examining systems of regulation by means of their frequency characteristics. The general differential equations of regulating systems are studied, and from them the relations between characteristics of a connected or disconnected system are deduced. The assignment of one of the frequency characteristics of a regulating system in a disconnected or connected state, or the assignment of its amplitude-phase characteristic is shown to be sufficient definition of the behavior of the system in the regulating process. Frequency characteristics for cases of nonzero starting conditions and a y disturbing force are considered.

Frequency criteria, given in conclusion, permit determination of the conditions which are necessary and sufficient for the regulating process to be accomplished without overregulation, or so that it may be affected monotonically. Appendices explain the method for finding, by means of an oscillogram of the transient process, the amplitude-phase, or frequency characteristics. Relations are derived to show that the mere assignment of a real frequency characteristic determines the assignment of an imaginary frequency characteristic, and vice versa. The same relations are deduced between the natural logarithm of an amplitude characteristic and a phase characteristic

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A. Differential Equations of a Regulation System in a State of Contact

Differential equations for any regulating system (see Figure 1, also the following system) can be expressed in the following form [1]:

$$\text{Equation of deviation} \quad \delta(t) = \theta(t) - \varphi(t), \quad (1)$$

$$\text{Equation of regulation} \quad R(p)\varphi = K_0 N(p)\delta, \quad (2)$$

$$\text{Equation of the object regulated} \quad D(p)\varphi = M(p)f(t) + L(p)\varphi, \quad (3)$$

where $p = \frac{d}{dt}$ and $\theta(t)$ is the required law of variation with time of the regulated magnitude; φ is the actual variation with time of the regulated magnitude; δ is the difference of the actual variation of the regulated magnitude and the specified law of variation; φ is the regulating action of magnitude at the output of the regulator; $f(t)$ is the disturbing force; $D(p)$, $M(p)$ and $L(p)$ are operators, depending only on the parameters of the regulated object; $R(p)$ and $N(p)$ are operators dependent only upon the parameters of the regulator; K_0 is the transmission ratio, or total amplification factor, of the regulator.

In the joint system of equations (1), (2), and (3), there are included three unknown functions φ , δ and φ and two functions $f(t)$ and $\theta(t)$, representing certain functions of time.

Eliminating from equations (1), (2), and (3) the functions φ and φ , we obtain the heterogeneous differential equation

$$[D(p)R(p) + K_0 N(p)L(p)]\delta = D(p)R(p)\theta(t) - M(p)R(p)f(t), \quad (4)$$

the solution of which represents the variation with time of deviation δ of the regulated magnitude, when the action of the two disturbances $\theta(t)$ and $f(t)$ is simultaneous.

In exactly the same way we can obtain from (1), (2), and (3), a differential equation for the regulated magnitude

$$[D(p)R(p) + K_0 N(p)L(p)]\varphi = M(p)R(p)f(t) + K_0 N(p)L(p)\theta(t). \quad (5)$$

The characteristic equation of the regulating system, as is clear from (4) or (5), will have the form:

$$D(\lambda)R(\lambda) + K_0 N(\lambda)L(\lambda) = 0. \quad (6)$$

Furthermore, for the sake of simplification, we shall assume that the regulating system is affected not by the two disturbing forces $\theta(t)$ and $f(t)$, but by either one of them. This supposition does not limit the statement as to general application, since, if the method of determining the effect produced by any of these disturbances separately is known, their joint effect may be found by making use of the principle of superposition.

Here it must be noted that, in the case of regulating systems, the task of which consists of maintaining the regulated magnitude at an assigned constant value, the disturbing force $\theta(t)$ is naturally absent, since in this case

$$\theta(t) = \theta_0 = \varphi_0 = \text{const},$$

where φ_0 represents the value of the regulated magnitude in the state of equilibrium of the system under consideration, from which an effort is being made to divert out-of-disturbances.

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On the contrary, in the case of servosystems, the force $\theta(t)$, which is received by the system from a given element and which in this case generally represents an arbitrary time function, is a basic disturbing force.

Furthermore, for the sake of a general statement, but on the basis of that made above, we assume in each case

$$f(t) = 0$$

that the force $\theta(t)$ from the given element, representing a given function of time, is a single disturbing force.

Now, assuming that in (4) and (5) $f(t) = 0$, we have

$$[D(p)R(p) + K_0N(p)L(p)] \delta = D(p)R(p) \theta(t), \quad (7)$$

$$[D(p)R(p) + K_0N(p)L(p)] \varphi = K_0N(p)L(p) \theta(t) \quad (8)$$

B. Differential Equations of a Regulating System in a Disconnected State

If the regulating system is disconnected, as shown in Figure 2, and an outside disturbing force $u(t)$ is introduced at the side of the break, the only change produced thereby in equations (1), (2), and (3) consists of replacing in the right-hand member of equation (2) the unknown variable δ by the assigned time function $u(t)$. The rest of the differential equation is not changed, since we assume that the regulator does not produce an inverse reaction on the object regulated and, therefore, equation (3) of the regulated object remains the same regardless of whether or not the regulator is connected with it.

Now, assuming, as in the case of the connected system, that $f(t) = 0$, we shall obtain

$$D(p)R(p) \delta = D(p)R(p) \theta(t) - K_0N(p)L(p)u(t), \quad (9)$$

$$D(p)R(p) \varphi = K_0N(p)L(p)u(t) \quad (10)$$

or

$$\varphi = K_0W(p)u(t), \quad (10a)$$

where

$$K_0W(p) = K_0 \frac{N(p)L(p)}{R(p)D(p)}, \quad (11)$$

Consequently, the characteristic equation of a regulating system in a disconnected state, as is clear from (9) or (10), has the form

$$D(\lambda)R(\lambda) = 0. \quad (12)$$

The operator $K_0W(p)$ may be called the operator of the regulating system in a disconnected state.

It is now easy to see that the characteristic equation (6) of a regulating system in a connected state consists of two basic components, the first of which, $D(\lambda)R(\lambda)$, represents the left side of the characteristic equation (12) of a regulating system in a disconnected state, and the second, $K_0N(\lambda)L(\lambda)$, determines those changes in characteristic equation (12) which cause the closing of the previously disconnected system. Moreover, we see from (11) that the operator of the disconnected system of regulation $K_0W(p)$ represents the ratio of the second of these components to the first (substituting in them λ for the operator p).

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C. Characteristic Function and Amplitude-Phase Characteristic of a Regulating System

As we know (see [1] for example) in order to analyze the stability of a regulating system with the help of Nyquist's criterion, there is introduced into the examination the function

$$K_0 W(z) = K_0 \frac{N(z)L(z)}{R(z)D(z)}, \quad (13)$$

the form of which is determined by the operator of the regulating system (11), considered as a function of the complex variable z .

The stability or instability of the regulating system is judged by the character of the curve described by the end of the characteristic vector

$$K_0 W(j\omega) = K_0 \frac{N(j\omega)L(j\omega)}{R(j\omega)D(j\omega)} \quad (13a)$$

with respect to the point $(-1, j0)$, or in other words, in accordance with the behavior of the function $K_0 W(z)$ along the imaginary axis.

The curve described by the end of the vector $K_0 W(j\omega)$ in the change of ω from $+\infty$ to $-\infty$, is called the amplitude-phase characteristic.

As will be shown below, the behavior of the characteristic function $K_0 W(z)$ along the imaginary axis permits not only judging the stability but also giving the complete characteristic of the dynamic properties of the regulating system [12].

In view of the special importance for us of the amplitude-phase characteristic $K_0 W(j\omega)$, let us pause first of all to explain its physical meaning.

Let us examine the regulating system in a disconnected state and find the induced oscillations $\phi_B(t)$, caused in it by the disturbing force $u(t)$ which represents a harmonic function of the form

$$\left. \begin{aligned} u(t) &= u_0 \cos \omega t \\ \text{or} \\ u(t) &= \frac{u_0}{2} e^{j\omega t} + \frac{u_0}{2} e^{-j\omega t} \end{aligned} \right\} \quad (14)$$

It is easily seen that, in accordance with (10) or (10a),

$$\left. \begin{aligned} \frac{\phi_B(t)}{u_0} &= \frac{1}{2} K_0 W(j\omega) e^{j\omega t} + \frac{1}{2} K_0 W(-j\omega) e^{-j\omega t} \\ \text{or} \\ \frac{\phi_B(t)}{u_0} &= K_0 H(\omega) \cos[\omega t - G(\omega)] \end{aligned} \right\} \quad (15)$$

where $H(\omega)$ and $G(\omega)$ represent, respectively, the modulus and argument of the expression $W(j\omega)$, i.e.,

$$W(j\omega) = H(\omega) e^{-jG(\omega)}, \quad (16)$$

or if in the expression for $W(j\omega)$, we single out the real part $U(\omega)$ from the imaginary $V(\omega)$:

$$W(j\omega) = U(\omega) + jV(\omega), \quad (17)$$

we may write

$$\left. \begin{aligned} H(\omega) &= \sqrt{U^2(\omega) + V^2(\omega)}, \\ G(\omega) &= \arctan \left[\frac{V(\omega)}{U(\omega)} \right] \end{aligned} \right\} \quad (18)$$

The function $H(\omega)$ and $G(\omega)$ are called amplitude and phase characteristics, respectively, and the functions $U(\omega)$ and $V(\omega)$ are called the real and imaginary frequency characteristics of a regulating system in a disconnected state.

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Thus we see that the assignment of the characteristic function $K_0 W(j\omega)$ permits defining the relative amplitude and phase of the induced harmonic oscillation in a disconnected regulating system at any frequency ω . Hence the function $K_0 W(j\omega)$, by analogy with the corresponding function encountered in the theory of electric circuits, may be considered as complex conductivity (in a generalized sense) of a regulating system in a disconnected state.

Thus, it is obvious that the amplitude-phase characteristic of a regulating system may be determined not only from its differential equations but also by experiment (See [1], and Appendix I).

As an explanatory example of the manner in which the amplitude-phase and frequency characteristics may be found from their differential equations, let us examine the servosystem represented in Figure 3. It consists of the master axis 1, the variation with time in the deflection angle of which is $\theta(t)$, a servoaxis 2 set in motion by the electric motor 3, a meter 4 recording the angle of discordance or deviation of the regulated magnitude δ , an element 5 which gives a signal proportional to the rate of change $\frac{d\delta}{dt}$ of the angle of deviation, an electronic amplifier 6, at the input of which a signal is given equal to the sum of the signals from both measuring elements 4 and 5, and excitation coil 7, and a generator 8 feeding the power motor 3 according to the Ward-Leonard system.

If it be assumed that the element 5 has its own moment of inertia, elasticity, and tensile friction, (for example, a precessional gyroscope) and that it has a critical attenuation, the connection between the angle of deviation δ and the voltage u at the input of the amplifier may be represented in the form:

$$u = n_0 \left[1 + \frac{\nu_1 p}{(1 + \tau_1 p)^2} \right] \delta,$$

where τ_1 is the time constant of the "predicted" element 5 giving the first derivative, n_0 the constant of the meter of the angle of deviation, and ν , the constant of the predicted element 5.

Designating a the grid-plate transconductance of the characteristic of the amplifier, i the variation of the excitation current of the generator, and L_p and R_p the inductance and ohmic resistance of the excitation coil, we may write the equation of the circuit of the armatures of the generator 8 and motor 3 in the following form:

$$L_a \frac{di_a}{dt} + R_a i_a + a \frac{d\varphi}{dt} = k_b i_p,$$

where i_a is the variation of the armature current; L_a and R_a , the total inductance and the ohmic resistance of the armatures of the generator and motor; a , the electromagnetic constant of the motor; k_b , the tangent of the angle of inclination of the characteristic of the motor.

The equation of the servoaxis, if it be assumed that it has only inertia, and if friction be disregarded, will be $J \frac{d^2\varphi}{dt^2} = k_a i_a$, where J is the moment of inertia.

Thus the interdependence of the angle of deviation δ and the angle of the servoaxis φ may be represented in the form:

$$\delta = K_n \frac{1 + \frac{\nu p}{1 + \tau_1 p^2}}{p(T_p^2 p^2 + 2\tau_a T_p + 1)(T_b p + 1)} \varphi,$$

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where K_0 is the amplification factor (or gear ratio of the system) and equals

$$K_0 = n_0 \cdot \frac{s}{R_b} \cdot \frac{L_b}{a}, T_a = \sqrt{\frac{J L_a}{R_a}}, \zeta_a = \frac{R_a}{2} \sqrt{\frac{J R_a}{a}}, T_b = \frac{L_b}{R_b},$$

or

$$\delta = K_0 W(p) \varphi,$$

where

$$W(p) = \frac{(1 + \tau_1 p)^2 + \nu_1 p}{p(\tau_a^2 p^2 + 2\zeta_a T_a p + 1)(T_b p + 1)(1 + \tau_1 p)^2},$$

and, consequently, the expression for the amplitude-phase characteristic $K_0 W(j\omega)$ of the servosystem will have the form

$$K_0 W(j\omega) = K_0 \frac{(1 + \tau_1 j\omega)^2 + \nu_1 j\omega}{j\omega(1 - \tau_a^2 \omega^2 + 2\zeta_a T_a j\omega)(T_b j\omega + 1)(1 + \tau_1 j\omega)^2} \quad (19)$$

The amplitude-phase characteristic corresponding to (19) when

$$K = 12, \tau_1 = 0.01, \nu_1 = 0.15, T_a = 0.07, \zeta_a = 1.6, T_b = 0.1,$$

is shown in Figures 4 and 4a (in Figure 4a part of the same characteristic as in Figure 4 is shown on a larger scale). Frequency characteristics, found from (19), in accordance with formulas (17) and (18) are represented in Figure 5.

D. Induced Harmonic Oscillations and Frequency Characteristics of a Connected Regulating System

Let us now consider the induced harmonic oscillations $\delta_B(t)$ and $\varphi_B(t)$ in a connected regulating system.

Obtaining in (9) and (10)

$$\left. \begin{aligned} \theta(t) &= q_0 \cos \omega t, \\ \text{or} \quad \theta(t) &= \frac{\theta_0}{2} e^{j\omega t} + \frac{\theta_0}{2} e^{-j\omega t}, \end{aligned} \right\} \quad (20)$$

we shall obtain:

$$\left. \begin{aligned} \frac{\delta B(t)}{\theta_0} &= \frac{1}{2} \Phi_B(j\omega) e^{j\omega t} + \frac{1}{2} \Phi_B(-j\omega) e^{-j\omega t} \\ \text{and} \quad \frac{\varphi_B(t)}{\theta_0} &= \frac{1}{2} \Phi(j\omega) e^{j\omega t} + \frac{1}{2} \Phi(-j\omega) e^{-j\omega t}, \end{aligned} \right\} \quad (21)$$

where

$$\Phi_B(j\omega) = \frac{D(j\omega) R(j\omega)}{D(j\omega) R(j\omega) + K_0 N(j\omega) L(j\omega)} \quad (22)$$

and

$$\Phi(j\omega) = \frac{K_0 W(j\omega) L(j\omega)}{D(j\omega) R(j\omega) + K_0 N(j\omega) L(j\omega)}, \quad (23)$$

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or, taking (13) into consideration,

$$\Phi_S(j\omega) = \frac{1}{1+K_0W(j\omega)} \quad (24)$$

and

$$\Phi(j\omega) = \frac{K_0W(j\omega)}{1+K_0W(j\omega)} \quad (25)$$

Hence, if we put

$$\Phi_S(j\omega) = P_S(\omega) + jQ_S(\omega) = A_S(\omega)e^{-jB_S(\omega)} \quad (26)$$

$$\Phi(j\omega) = P(\omega) + jQ(\omega) \quad (27a)$$

and

$$\Phi(j\omega) = A(\omega)e^{-jB(\omega)} \quad (27b)$$

the induced oscillations in a connected system will be determined by the expressions

$$\left. \begin{aligned} \frac{\delta B(t)}{\delta_0} &= A_S(\omega) \cos[\omega t - B_S(\omega)] \\ \frac{\delta B(t)}{\delta_0} &= A(\omega) \cos[\omega t - B(\omega)] \end{aligned} \right\} \quad (28)$$

The functions $\Phi_S(j\omega)$ and $\Phi(j\omega)$, by analogy with the theory of electric circuits, may be called complex conductivities of a system in a connected state.

It is evident that the frequency characteristics P_S , Q_S , A_S , B_S and P , Q , A and B of a system in a connected state are connected with one another in the same relations as the frequency characteristics U , V , H , and G of a system in a disconnected state. See (17).

E. Determination of the Frequency Characteristics of a Connected System According to a Given Amplitude-Phase Characteristic

On the basis of formulas (24) and (25) it is easy to see that the real $P_S(\omega)$, $P(\omega)$ and the imaginary $Q_S(\omega)$, $Q(\omega)$ frequency characteristics of a connected system can be determined according to the frequency characteristics $U(\omega)$, $V(\omega)$ of a disconnected system, which, in their turn, may be determined according to the given amplitude-phase characteristic $K_0W(j\omega)$, see (16), with the help of the formulas

$$\left. \begin{aligned} P_S(\omega) &= \frac{1+K_0U(\omega)}{[1+K_0U(\omega)]^2+K_0^2V^2(\omega)} = \frac{1+K_0H\cos G}{1+2K_0H\cos G+K_0^2H^2} \\ Q_S(\omega) &= \frac{-K_0V(\omega)}{[1+K_0U(\omega)]^2+K_0^2V^2(\omega)} = \frac{-K_0H}{1+2K_0H\cos G+K_0^2H^2} \end{aligned} \right\} \quad (29)$$

$$\left. \begin{aligned} P(\omega) &= \frac{K_0U(\omega)[1+K_0U(\omega)]+K_0^2V^2(\omega)}{[1+K_0U(\omega)]^2+K_0^2V^2(\omega)} = \frac{K_0H\cos G+K_0^2H^2}{1+2K_0H\cos G+K_0^2H^2} \\ Q(\omega) &= \frac{K_0V(\omega)}{[1+K_0U(\omega)]^2+K_0^2V^2(\omega)} = \frac{-K_0H\sin G}{1+2K_0H\cos G+K_0^2H^2} \end{aligned} \right\} \quad (30)$$

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In regard to formulas (29) and (30) the following should be noted:

In Appendix II, it is shown that, as many authors have proved, (see [5], [17]), the real frequency characteristic of the system may be found if the imaginary frequency characteristic is given and vice versa. (This statement is used for every real stable system with any finite number of degrees of freedom.)

The same statement may also be made in regard to the amplitude and phase characteristics. On the other hand, however, formulas (15) show that the amplitude and phase characteristics may always be found through the given real and imaginary frequency characteristic, and vice versa.

Thus we arrive at the following conclusion: in order to determine all frequency characteristics of a system, both in a connected or disconnected state, (with a fixed disturbing force and given starting conditions), it is sufficient to give any one of its frequency characteristics, either in a connected or disconnected state.

Hence it follows that the assignment of a form of any one of the frequency characteristics determines the form of the remainder.

The frequency characteristics of a connected system can be very simply determined graphically through the amplitude-phase characteristic (2).

The modulus of $\Phi_S(j\omega)$ at a given value ω will be equal (see Figure 6) to the ratio of the lengths of the vectors \overline{KO} and \overline{KL} , that is,

$$A_S(\omega) = |\Phi_S(j\omega)| = \frac{1}{|1 + K_0 W(j\omega)|} = \frac{|\overline{KO}|}{|\overline{KL}|}, \quad (31)$$

and the argument (or phase) will be equal to the angle in radians between these same vectors, that is,

$$B_S(\omega) = \text{arc } OKL. \quad (32)$$

Just so

$$A(\omega) = |\Phi(j\omega)| = \frac{|K_0 W(j\omega)|}{|1 + K_0 W(j\omega)|} = \frac{|\overline{OL}|}{|\overline{KL}|}, \quad (33)$$

$$B(\omega) = |\Phi(j\omega)| = \text{arc } OLK. \quad (34)$$

Thus, the frequency characteristics of a system in a connected state can be determined analytically or graphically through a given amplitude-phase characteristic or through the frequency characteristic of the disconnected system.

For the example examined above, in conformity with (19), (24), and (25), the expressions for $\Phi_S(j\omega)$ and $\Phi(j\omega)$, will obviously have the form:

$$\Phi_S(j\omega) = \frac{j\omega [T_a^2(j\omega)^2 + 2\zeta_a T_a j\omega + 1] (T_b j\omega + 1) [1 + \tau_1(j\omega)]^2}{j\omega [T_a^2(j\omega)^2 + 2\zeta_a T_a j\omega + 1] (T_b j\omega + 1) (\tau_1 j\omega + 1)^2 + K_0 [\tau_1 j\omega + 1]^2 + \nu_1 j\omega}$$

and

$$\Phi(j\omega) = \frac{K_0 [\tau_1 j\omega + 1]^2 + \nu_1 j\omega}{[j\omega [T_a^2(j\omega)^2 + 2\zeta_a T_a j\omega + 1] (T_b j\omega + 1) (\tau_1 j\omega + 1)^2 + K_0 [\tau_1 j\omega + 1]^2 + \nu_1 j\omega}. \quad (35)$$

The frequency characteristics P, Q, A, B, which may be found directly with the help of (35) or graphically from Figure 4 or Figure 5 according to the rule given above, are depicted in Figure 7.

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F. Representing the Functions $\theta(t)$ and $\varphi(t)$, Which Determine the Regulating Process, in the Form of Fourier Integrals

Let us now assume that, for the regulating system in a connected state which we are examining, there is applied not the harmonic disturbing force (19) but the disturbance $\theta(t)$, which is of arbitrary form but satisfies the conditions which must be observed so that it can be represented in the form of a Fourier integral [3]:

$$\theta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(j\omega) e^{-j\omega t} d\omega, \quad (36)$$

where

$$\psi(j\omega) = \int_0^{\infty} \theta(t) e^{-j\omega t} dt. \quad (37)$$

In (37) the lower limit is taken as 0, since we are supposing that $\theta(t) = 0$ when $t < 0$.

Now, in accordance with (36), a disturbing force in the general case may be considered the limit of an infinite sum of harmonic oscillations of the form

$$\frac{1}{2\pi} \psi(j\omega) d\omega e^{j\omega t}$$

each of which, according to the foregoing, causes the induced oscillations:

$$\Phi_s(j\omega) \frac{1}{2\pi} \psi(j\omega) d\omega e^{j\omega t}.$$

Since we are examining linear systems, in which the principle of superposition holds, the effect produced by the disturbing force $\theta(t)$ will thus be equal to

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(j\omega) \Phi_s(j\omega) e^{j\omega t} d\omega, \quad (38)$$

and in just the same way

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(j\omega) \Phi(j\omega) e^{j\omega t} d\omega. \quad (39)$$

The function $\psi(j\omega)$ is called the complex frequency spectrum of the disturbing force $\theta(t)$.

Assuming that

$$\psi(j\omega) = F(\omega) e^{jC(\omega)} \quad (40)$$

and taking (26) and (27) into consideration, instead of (38) we can write:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) A_s(\omega) e^{j[\omega t + C(\omega) - B_s(\omega)]} d\omega, \quad (41)$$

and in like manner

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) A(\omega) e^{j[\omega t + C(\omega) - B(\omega)]} d\omega, \quad (42)$$

whence it follows that the transient process with any disturbing force is determined by the assignment of the amplitude $F(\omega)$ and the phase $C(\omega)$ of the frequency spectra of the disturbing force and also by the amplitude A_s (or A) and phase B_s (or B) of the frequency characteristics of the system under consideration.

In analyzing the dynamics of regulating systems in the nature of the effect $\theta(t)$, frequent use is made of the so-called unit function of Heaviside which we shall express as [1]; it is defined thus:

$$[1] = \begin{cases} = 1 & \text{when } t > 0, \\ = 0 & \text{when } t < 0. \end{cases} \quad (43)$$

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Now, let us assume that

$$\text{where } \begin{cases} \theta(t) = \theta_0 I(t), \\ \theta_0 = \text{const.} \end{cases} \quad (44)$$

The function (44) obviously does not satisfy the conditions of the Fourier integral theorem, since

$$\int_{-\infty}^{+\infty} |\theta(t)| dt \quad (45)$$

does not exist for it and, consequently, it is not possible to use formulas (36) to (39). In order to surmount this difficulty, it is possible to change this function so that it will satisfy all the conditions of the Fourier integral theorem and at the same time have a form sufficiently close to the required form.

It is possible to do this by several methods [4]. One of the methods is included in the following:

$$\text{Let } \begin{cases} \theta(t) = 0 & \text{when } t < 0, \\ \theta(t) = \theta_0 e^{-ct} & \text{when } t > 0, \end{cases} \quad (46)$$

where c is a small, real, positive quantity.

It is evident that, by selecting a sufficiently small quantity for c , we shall always be able to manage so that the functions (44) and (46) will differ one from the other less than by an previously assigned constant magnitude ϵ within the limits of any finite interval of time t . Function (46) satisfies the conditions of the Fourier integral theorem, and hence, we can make use of the above-stated formulas for it. Furthermore, we perceive that this will remain true even if $c \rightarrow 0$, with the proper interpretation of this limit.

In fact, by substituting (46) and (37), we shall obtain

$$\Psi(j\omega) = \theta_0 \int_0^{\infty} e^{-(c+j\omega)t} dt = \frac{\theta_0}{c+j\omega}. \quad (47)$$

Thus, in accordance with (36), the function (46) may be represented in the following manner:

$$\theta(t) = \frac{\theta_0}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{c+j\omega} d\omega. \quad (48)$$

The time function (44) is the limit of this expression when $c \rightarrow 0$. If we simply assume that $c = 0$, the integral becomes improper, as then the expression behind the integral sign will have a pole at the origin of coordinates, through which passes the path of integration which is the imaginary axis. Thus, if we wish to make $c = 0$, it is necessary to alter the path of integration with the aid of a small semicircle at the origin of the coordinates (Figure 8).

As the Fourier transformation is true not only for imaginary but also for complex values of the argument, we can assume in (48) that $j\omega = z$, where z is a complex variable.

Then, instead of (48), we shall obtain

$$\theta(t) = \frac{\theta_0}{2\pi j} \int_{-\infty}^{+\infty} \frac{e^{zt}}{z} dz, \quad (49)$$

where the integral is taken along the contour in Figure 8.

Let us suppose now that the disturbing force in the form (48) is applied to the system with complex conductivity $\Phi(j\omega)$. Then, according to (39), we can write:

$$\frac{\varphi(t)}{\theta_0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(j\omega)}{c+j\omega} e^{j\omega t} d\omega. \quad (50)$$

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Assuming that $1/z = z$ and selecting as the path of integration the contour in Figure 8, we can assume, just as above, that $c = 0$, and then

$$\frac{\varphi(t)}{\theta_0} = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\Phi(z)}{z} e^{zt} dz \quad (51)$$

We shall now express the transient process, caused in the system by a single disturbing influence (44) through its frequency characteristics.

G. Expressions Determining the Functions $\delta(t)$ and $\varphi(t)$ Through the Frequency Characteristics of the System

As, in accordance with (49),

$$\Phi(0)[1] = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\Phi(z)}{z} e^{zt} dz, \quad (52)$$

instead of (51) we can write

$$\frac{\varphi(t)}{\theta_0} - \Phi(0)[1] = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{\Phi(z) - \Phi(0)}{z} e^{zt} dz. \quad (53)$$

As the expression behind the integral sign does not now have a singular point at the origin of the coordinates, assuming that

$$\left. \begin{aligned} z = j\omega \\ \Phi(z) = P(\omega) + j Q(\omega), \\ e^{zt} = \cos \omega t + j \sin \omega t, \end{aligned} \right\} \quad (54)$$

we can proceed from the complex variable z to the real variable ω . Further, it is necessary to make use of the following properties of the function $\Phi(z)$. For physical reasons it is evident that the mathematical expression for this function can have only real coefficients.

Hence, the expressions $\Phi(j\omega)$ and $\Phi(-j\omega)$ are complex and conjugate, that is,

$$\left. \begin{aligned} P(\omega) &= P(-\omega), \\ Q(\omega) &= -Q(-\omega). \end{aligned} \right\} \quad (55)$$

Consequently, the function $P(\omega)$ is an even function and $Q(\omega)$ an odd function of ω , and hence

$$\left. \begin{aligned} P(0) &= \Phi(0), \\ Q(0) &= 0. \end{aligned} \right\} \quad (56)$$

Substituting (54) and (55) and taking into account (55), after some conversions (see [5] or [6]) we shall obtain

$$\frac{\varphi(t)}{\theta_0} = \frac{2}{\pi} \int_0^{\infty} \frac{P(\omega)}{\omega} \sin t\omega d\omega \quad (57)$$

or

$$\frac{\varphi(t)}{\theta_0} = P(0) + \frac{2}{\pi} \int_0^{\infty} \frac{Q(\omega)}{\omega} \cos t\omega d\omega \quad (57a)$$

and similarly

$$\frac{\delta(t)}{\theta_0} = \frac{2}{\pi} \int_0^{\infty} \frac{P\delta(\omega)}{\omega} \sin t\omega d\omega. \quad (58)$$

On the basis of (57) or (57a), and taking into account (30) or (31), we can express the transient process produced in a connected system by a single disturbing force through its frequency characteristics $U(\omega)$ and $V(\omega)$ in a disconnected state with the help of the formulas

$$\frac{\varphi(t)}{\theta_0} = \frac{2}{\pi} \int_0^{\infty} \left\{ \frac{K_a U(\omega) [1 + K_a U(\omega)]}{[1 + K_a U(\omega)]^2 + K_o^2 V^2(\omega)} \right\} \frac{\sin t\omega}{\omega} d\omega \quad (59)$$

and

$$\frac{\varphi(t)}{\theta_0} = P(0) + \frac{2}{\pi} \int_0^{\infty} \left\{ \frac{K_a V(\omega)}{[1 + K_a U(\omega)]^2 + K_o^2 V^2(\omega)} \right\} \frac{\cos t\omega}{\omega} d\omega. \quad (59a)$$

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Formulas (57) and (57a), or (59) and (59a), show that the transient process in a regulating system with a single disturbing force is fully determined by the assignment of its frequency characteristics in a connected or disconnected state [13].

Thus we reach an important conclusion: the assignment of the characteristic function $K_W(z)$ or of the amplitude-phase characteristic $K_W(j\omega)$ permits not only analyzing the stability and determining the induced harmonic oscillations in the system but also determining the picture of the transient process; that is, the assignment of a characteristic function $K_W(z)$ is sufficient, in the case of a disturbing force in the form of a Heaviside unit function and under zero starting conditions, for a complete characteristic of the dynamic properties of a system from the viewpoint of the condition of stability and of the condition of the quality of regulation.

H. Application of a Method of Frequency Characteristics in Case of Any Disturbing Influence and of Nonzero Starting Conditions

(The following generalization of a method of frequency characteristics, as far as is known to us, is being given for the first time.)

The expressions (57) and (58) determine the regulating process through one of the frequency characteristics of a regulating system in a connected state with a disturbing force in the form of a Heaviside unit function and with zero starting conditions.

In fact, as we already know [12], the transient process $\varphi(t)$ in a linear dynamic system affected by a disturbing force $f(t)$, with some limitations [12], may be determined by the aid of the integral

$$\varphi(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} J(z) e^{zt} dz, \quad (60)$$

where

$$J(z) = \Psi(z) \Phi(z) + \Psi_H \Phi_0(z). \quad (61)$$

The functions $\Phi(z)$, $\Phi_0(z)$ may be determined from the differential equations of the system, and the function $\Phi(z)$ in the case of regulating systems is determined by formula (25), or (24), and the function $\Phi_0(z)$ is determined by the expression

$$\Phi_0(z) = \frac{1}{D(z)R(z) + K_0 N(z)L(z)}.$$

The function $\Psi(z)$ is determined by Laplace's transformation for a disturbing force $f(t)$, that is,

$$\Psi(z) = \int_0^{\infty} e^{-zt} f(t) dt, \quad (62)$$

and in the particular case when $f(t) = [1]$, it takes the form

$$\Psi(z) = \frac{1}{z}, \quad (63)$$

and the function $\Psi_H(z)$, in the case of systems with centered parameters, may be determined from the differential equation of the system and from initial conditions.

The rule for finding the function $\Psi_H(z)$ is included in the following. Let us assume that the equation of the system takes the form

$$a_n \frac{d^n \varphi}{dt^n} + a_{n-1} \frac{d^{n-1} \varphi}{dt^{n-1}} + \dots + a_1 \frac{d\varphi}{dt} + a_0 \varphi = f(t),$$

and when $t=0$

$$\varphi = \varphi_0, \quad \frac{d\varphi}{dt} = \varphi_1, \quad \frac{d^2 \varphi}{dt^2} = \varphi_2, \dots, \quad \frac{d^{n-1} \varphi}{dt^{n-1}} = \varphi_{n-1}.$$

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Then, to find $\Psi_H(z)$, it is necessary to make a table

	z^{n-1}	z^{n-2}	z^{n-3}	...	1
ϕ_0	a_n	a_{n-1}	a_{n-2}	...	a_1
ϕ_1	0	a_n	a_{n-1}	...	a_2
ϕ_2	0	0	a_n	...	a_3
...
ϕ_{n-1}	0	0	0	...	a_n

and to multiply each of the coefficients of all the lines of this table by z to the same power that z has in this column, and by ϕ with the same sign that this magnitude has in the same column as the coefficient a_k under consideration, and then add the results, that is,

$$\begin{aligned}\Psi_H(z) = & \phi_0 (a_n z^{n-1} + a_{n-1} z^{n-2} + \dots + a_1) + \\ & + \phi_1 (a_n z^{n-2} + a_{n-1} z^{n-3} + \dots + a_2) + \\ & + \phi_2 (a_n z^{n-3} + a_{n-1} z^{n-4} + \dots + a_3) + \\ & \dots + \phi_{n-1} a_n.\end{aligned}$$

If the function $J(z)$, made up by the method given above of $\Psi(z)$, $\Psi_H(z)$ and $\Phi(z)$, satisfies the conditions of the Fourier integral theorem, it is possible to put $c=0$ into formula (60) and to carry out integration along the imaginary axis; that is, we can write

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J(j\omega) e^{j\omega t} d\omega \quad (64)$$

hence, assuming that

$$\begin{aligned}J(j\omega) &= X(\omega) + jQ(\omega) \\ \Psi(j\omega) &= \gamma(\omega) + j\chi(\omega) \\ \Psi_H(j\omega) &= \gamma_H(\omega) + j\chi_H(\omega)\end{aligned} \quad (65)$$

.. and taking into account (61), we can write

$$\varphi(t) = \frac{1}{\pi} \int_0^{\infty} [X(\omega) \cos \omega t - Y(\omega) \sin \omega t] d\omega, \quad (66)$$

where the functions $X(\omega)$ and $Y(\omega)$, which we call, respectively, real and imaginary frequency functions or generalized frequency characteristics, can be expressed by the real frequency characteristic $P(\omega)$ and the imaginary frequency characteristic $Q(\omega)$ and by the real $\gamma(\omega)$, $\gamma_H(\omega)$ and the imaginary $\chi(\omega)$, $\chi_H(\omega)$ parts of the frequency spectrum $\Psi(\omega)$ of the disturbing force and the function $\Psi_H(\omega)$ from the initial conditions with the aid of the relations:

$$\begin{aligned}X &= \gamma P + \gamma_H P_0 - \chi Q - \chi_H Q_0 \\ Y &= \chi P + \chi_H P_0 + \gamma Q + \gamma_H Q_0\end{aligned} \quad (67)$$

If the function $J(z)$ has a pole at the origin of the coordinates, that is can be represented by the form $\frac{J_1(z)}{z}$ where $J_1(0) \neq 0$, it is more convenient, in this case, to define the respective real and imaginary parts of the expression $J_1(z)$, but not $J(z)$, through $X(\omega)$ and $Y(\omega)$.

By the same reasoning in this case as above, we have for the transient process expressions which will differ from (57), (57a), and (58) only in that the functions $X(\omega)$ and $Y(\omega)$ will be included in them instead of the functions $P(\omega)$ and $Q(\omega)$.

Thus, in this case, with any disturbing force and with the nonzero conditions under which the integral (64) exists, the regulating process will be determined through the function $X(\omega)$ and $Y(\omega)$ with the help of the relations:

$$\varphi(t) = \frac{2}{\pi} \int_0^{\infty} \frac{X(\omega)}{\omega} \sin \omega t d\omega, \quad (68)$$

or

$$\varphi(t) = X(0) + \frac{2}{\pi} \int_0^{\infty} \frac{Y(\omega)}{\omega} \cos \omega t d\omega, \quad (68a)$$

and similarly

$$\delta(t) = \frac{2}{\pi} \int_0^{\infty} \frac{X_0(\omega)}{\omega} \sin \omega t d\omega \quad (69)$$

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or

$$\delta(t) = X_s(0) + \frac{2}{\pi} \int_0^{\infty} \frac{Y_s(\omega)}{\omega} \cos \omega t d\omega \quad (69a)$$

I. Frequency Criteria of the Absence of Overregulation and Frequency Criteria of Monotony

Calculation of the integrals (57), (58), (68), and (69) is usually very complicated in cases which are of interest in practice. Hence the question arises: is it possible to judge the character of a transient process from the viewpoint of conditions in regulation, according to the form of the frequency characteristics $P(\omega)$, $Q(\omega)$ or $A(\omega)$, $B(\omega)$ without calculating these integrals?

We have already touched upon this question in part in [7]. Here we shall confine ourselves to an indication in regard to two properties, of great practical importance in our opinion, of the generalized real frequency function $X_S(\omega)$, which with zero initial conditions and with a disturbing force in the form of a Heaviside unit function, see (44), amounts to the real frequency characteristic $P(\omega)$.

Before doing this, however, we shall introduce some definitions. Let us examine the curve in Figure 9, characterizing the variation with time of the deviation of a regulated magnitude. Let us agree to call the regulating process oscillating, if in its course the deviation $\delta(t)$ at some moment of time is shown to be sometimes greater, sometimes less than the magnitude of the static deviation (curve I).

We shall say that the process does not have overregulation if the deviation $\delta(t)$ remains less than the magnitude of the static deflection δ_0 during the whole process (curve II).

We shall call the regulation process monotonic if the function $\delta(t)$ represents a monotonically increasing function of time tending toward the magnitude δ_0 when $t \rightarrow \infty$ (curve III).

From the definitions given, it is evident that all monotonic processes do not have overregulation, but that not every process without overregulation is monotonic (for instance, curve II in Figure 9).

Now let us examine the properties which must be possessed by the generalized real frequency functions $X_S(\omega)$: 1) so that the process may not have overregulation and 2) so that the process may be monotonic.

As we know from [3], if $X_S(\omega)$ is a positive, nonincreasing, continuous function and $X_S(\omega)$ is connected with the $\delta(t)$ relation of the form (68), the function $\delta(t)$ represents a positive function less than the constant quantity δ_0 whatever t may be, and $\delta_0 = \delta(\infty) = X_S(0)$.

Furthermore, it may be demonstrated that if $X_S(\omega)$ is a positive, monotonically decreasing function, then $\delta(t)$ represents a positive, monotonically increasing function and $\delta_0 = \delta(\infty) = X_S(0)$.

Thus, on the basis of the statements just made, we can make the following assertion, which may be called the frequency criterion of the absence of overregulation. (We have already stated this criterion, see [7], but in its application to a usual and not to a generalized frequency characteristic. The frequency criterion for monotonicity is presented for the first time.)

In order to bring about the regulating process without overregulation, it is necessary and sufficient that the real frequency function $X_S(\omega)$ of the regulating system in a connected state represent a positive, nonincreasing, continuous function of the frequency ω .

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In conclusion, we formulate the frequency criterion for monotonicity:

That the regulating process may be brought about monotonically, it is necessary and sufficient that the real frequency function $X_g(\omega)$ represent a positive, monotonically decreasing function of the frequency ω .

In view of their simplicity, the criteria formulated seem to have great practical significance.

Conclusions

1. The assignment of one of the frequency characteristics of a regulating system with a finite number of degrees of freedom in a disconnected state is sufficient to determine the form of its amplitude-phase characteristic.
2. The frequency characteristics of a regulating system in a connected state may be found through its frequency characteristics in a disconnected state both analytically and graphically.
3. The regulating process, induced by a single disturbing force (Heaviside unit function), is determined by the assignment of the amplitude-phase characteristic or of the frequency characteristics of a system in a disconnected or connected state.
4. The transient process in a stable regulating system, in case of any disturbing force and nonzero initial conditions, may be determined if one of the frequency characteristics of the system in a connected or disconnected state, the frequency spectrum of the disturbing force and the initial conditions are given.
5. To avoid overregulation, it is necessary and sufficient that the real frequency function $X_g(\omega)$ represent a positive, nonincreasing, continuous function of the frequency ω .
6. To effect the regulating process monotonically, it is necessary and sufficient that the real frequency function $X_g(\omega)$ represent a positive, continuous, monotonically decreasing function of the frequency ω .

Appendix I. Finding Amplitude-Phase Characteristics From an Oscillogram of the Transient Process

As we know, the usual experiment method of finding amplitude-phase characteristics comes down to the measurement of the amplitude and phase of the induced harmonic oscillations in a disconnected system at various frequencies.

However, finding the amplitude-phase characteristics experimentally by the method mentioned above may sometimes be inconvenient, especially in case of non-electric members with large time constants, because of the necessity for special apparatus for measuring the amplitude and phase of the induced harmonic oscillations at different frequencies.

At the same time, obtaining an oscillogram of a transient process in such members, in case of applying to their input a disturbing force of any determined form, often presents no significant experimental difficulties.

Hence, the question arises, is it possible to indicate a simple method of finding the amplitude-phase characteristics of a system directly from an oscillogram of the transient process induced in it by a disturbing force of any determined form?

It is demonstrated that this question may be answered in the affirmative.

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The method suggested below for finding the amplitude-phase characteristics of various members of servosystems or regulating systems by means of an oscillogram of the transient process was worked out by Bedford and Fredendall [8]. This method was used to determine amplitude and phase frequency characteristics of television apparatus with an oscillogram of the transient process induced in the apparatus by a disturbing force in the form of a rectangular impulse of sufficiently long duration in comparison with the duration of the transient process.

We may assume that it is not the rectangular impulse which is the disturbing force, but the single disturbance, that is, the disturbance arising at the moment of time $t = 0$, and lasting any desired time.

The basic hypothesis on which this method is founded consists in the assumption that it is possible to plot a graduated curve of approximately the same harmonic form as the given curve (Figure 10). It is evident that in approximating a given curve by aid of the graduated curve it is possible to obtain any desired degree of accuracy, by taking any number of graduations in the field, corresponding with the irregular part of the process.

We see from Figure 10 that the graduated curve has the components g, h, i , etc. Each of these rectangular components is identical in form with a disturbing force and, consequently contains all the harmonics contained by the disturbance. However, each of the rectangular components differs one from the other and from the disturbance, both as to "amplitude" and as to the amount of lag. Therefore, the harmonics of the various rectangular components are out of line with the harmonics of the disturbance at different phase angles depending upon their lag.

Consequently, the ratio of the amplitude of each harmonic of the disturbance to the amplitude of the harmonic of the graduated curve represents the sum of the vectors, and the length of each of these vectors is proportional to the height of the graduations g, h, i , etc. Their deviation angles may be determined from the amount of the lag of the corresponding rectangular components.

Figure 11a shows the vector structure required to find the amplitude and phase corresponding to the frequency ω_n . Each component vector turns at the same angle in relation to the preceding vector, equaling $\omega_n \Delta t$ as the rectangular components g, h, i, \dots are out of line, one with another, along the time axis for one and the same interval of time Δt .

The components m, n, p, r are negative. Consequently, the vectors m, n, p, r are also negative. The length of the segment OB representing the vector sum of the vectors g, h , etc., determines the comparative amplitude of the system, and the angle Φ the phase at the frequency ω_n . Figure 11b is drawn for a frequency equal to $2\omega_n$. Here the angle between subsequent vectors is equal to $2\omega_n \Delta t$.

To facilitate plotting the vectors special nomographs may be made to reduce to a minimum [8] the time required to plot a graph.

Appendix II. Interrelation Between the Frequency Characteristics of a System [9]

It follows from relations (56) and (57) that some relation must exist between the frequency characteristics $P(\omega)$ and $Q(\omega)$.

Let us find this relation. Differentiating (56) and (57) with respect to ω it is easy to see that:

$$\int_0^{\infty} Q(\omega) \sin t\omega d\omega = - \int_0^{\infty} P(\omega) \cos t\omega d\omega \quad (70)$$

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As $P(\omega)$ is an even function, we can write:

$$P(\omega) = \frac{2}{\pi} \int_0^{\infty} \cos \omega u \, du \int_0^{\infty} P(x) \cos ux \, dx, \quad (71)$$

or, taking (70) into account,

$$P(\omega) = -\frac{2}{\pi} \int_0^{\infty} \cos \omega u \, du \int_0^{\infty} Q(x) \sin ux \, dx. \quad (72)$$

Similarly

$$Q(\omega) = -\frac{2}{\pi} \int_0^{\infty} \sin \omega u \, du \int_0^{\infty} P(x) \cos ux \, dx. \quad (73)$$

Proceeding from (72) and (73), we can pass over [10] to Hilbert's transformation, according to which

$$\left. \begin{aligned} P(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Q(u)}{\omega - u} \, du, \\ Q(\omega) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(u)}{\omega - u} \, du. \end{aligned} \right\} \quad (74)$$

It is necessary to consider the integrals included in (74) as the chief values of the integrals.

Let us now examine the connection between the amplitude $A(\omega)$ and the phase $B(\omega)$ frequency characteristics of the system.

Taking the logarithm of (27b), we obtain:

$$\ln \Phi(j\omega) = \ln A(\omega) + j[-B(\omega)]. \quad (75)$$

Comparing (75) with (27a):

$$\Phi(j\omega) = P(\omega) + j Q(\omega) \quad (27a)$$

It is easy to see that the natural logarithm of the amplitude characteristic and the phase characteristic (with a minus sign) are in the same relations one to the other as the real $P(\omega)$ and the imaginary $Q(\omega)$ frequency characteristics. Hence, on the basis of analogous reasoning, it may be demonstrated that the natural logarithm of the modulus $A(\omega)$ of the complex conductivity $\Phi(j\omega)$ and its argument $-B(\omega)$ are connected one with the other by Hilbert's transformation, that is, analogously to (72) and (73):

$$\ln A(\omega) = \frac{1}{\pi} \int_0^{\infty} \cos \omega u \, du \int_0^{\infty} B(x) \sin ux \, dx, \quad (76)$$

and

$$B(\omega) = \frac{1}{\pi} \int_0^{\infty} \sin \omega u \, du \int_0^{\infty} \ln A(x) \cos ux \, dx, \quad (77)$$

or

$$\ln A(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{B(u)}{\omega - u} \, du \quad (78)$$

and

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln A(u)}{\omega - u} \, du$$

Expression (74) shows that the real frequency characteristic can be found if the imaginary frequency characteristic is given, and vice versa. In precisely the same way, in accordance with (78), assignment of the amplitude characteristic predetermines (in the case of systems with minimum phase displacement) the form of the phase characteristic, and vice versa.

Bode [11, 14] showed that the following relation can also be obtained between the characteristics of the phase $B(\omega)$ and the amplitude $A(\omega)$

$$B(\omega) = \frac{1}{\pi} \int_0^{\infty} \frac{dA(u)}{du} \cdot \ln \coth \frac{|u|}{2} \, du,$$

which, although it seems more complicated than expression (78), can be integrated with comparative ease.

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frequency, above, above all, that:

1. The value of the function $B(\omega)$ at any frequency ω_c is in proportion to the tabulated value of the derivative of the amplitude characteristic at all frequencies, plotted on a logarithmic scale.

2. The most substantial effect -- as the form of the function $\ln \coth \frac{W}{2}$ in Figure 13 above -- on the value $B(\omega_c)$ at a given frequency ω_c is exercised by the inclination of the amplitude characteristic near this frequency, and the effect of the inclination at more remote frequencies decreases proportionally to the logarithm of their distance from the frequency ω_c .

[Expanded figures follow.]

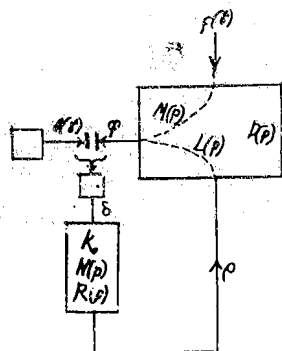


Figure 1. Regulating System in a State of Connect

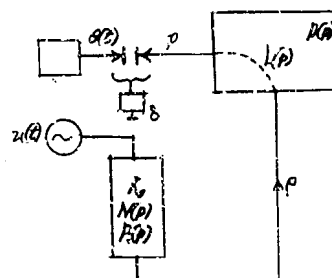


Figure 2. Regulating System in a Disconnected State

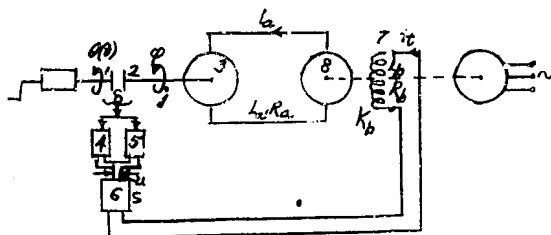


Figure 3. Structural Plan of the Servosystem

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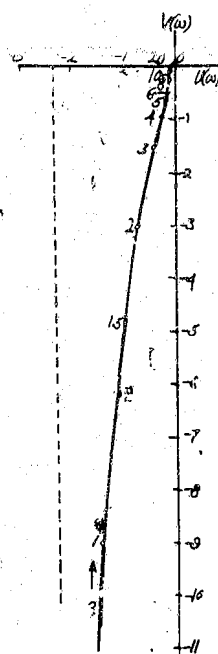


Figure 4. Amplitude-Phase Characteristic of the Servosystem

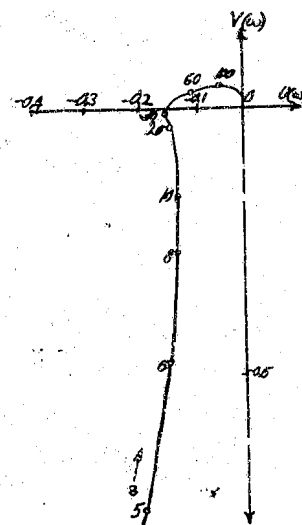


Figure 4a. Part of the Amplitude-Phase Characteristic of the Servosystem on a Larger Scale

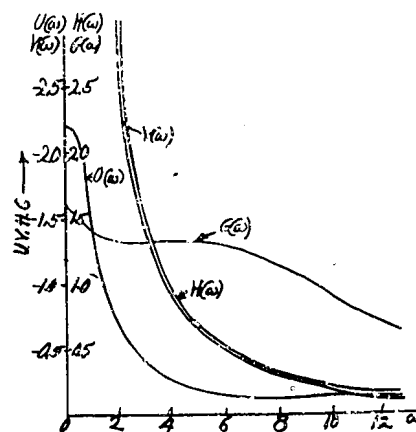


Figure 5. Frequency Characteristics of a Disconnected Servosystem:

$U(\omega)$ is Real, $V(\omega)$ is Imaginary,
 $X(\omega)$ is Amplitude, $G(\omega)$ is Phase

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Figure 6. Graphic Method of Finding the Frequency Characteristics of a Connected System by a Given Amplitude-Phase Characteristic

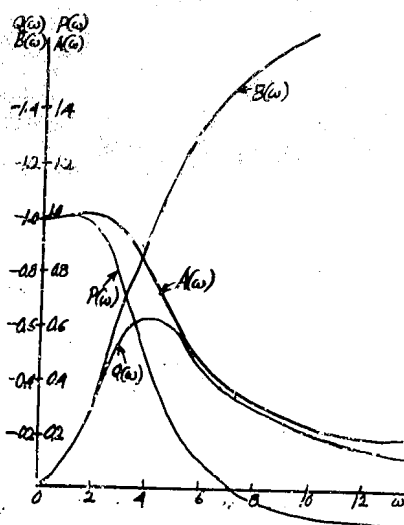
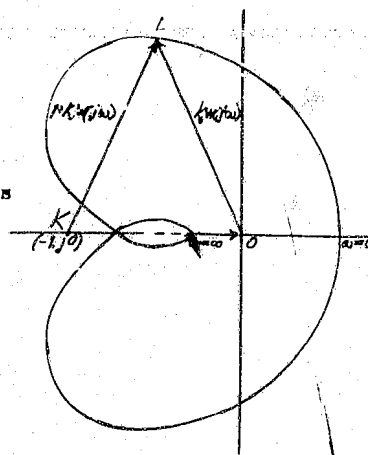
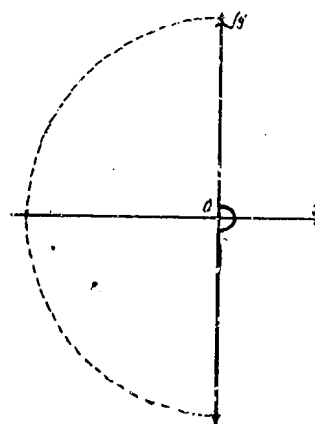


Figure 7. Frequency Characteristics of a Connected System

Figure 8. Path of Integration to Represent the Heaviside Unit Function in the Form of a Contour Integral



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Figure 10. Finding the Amplitude-Phase Characteristic by Means of an Oscillogram of the Transient Process

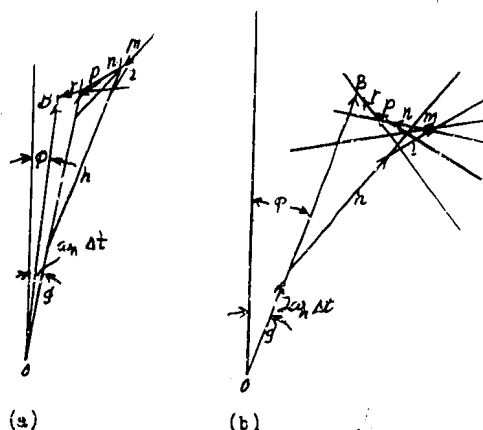
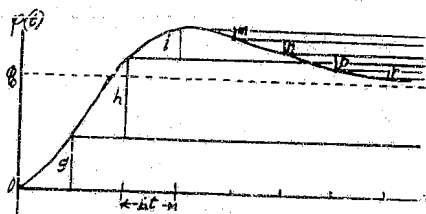


Figure 11a. Determination of the Modulus and Phase of the Vector $K_0 W(j\omega)$ at the Frequency ω_0

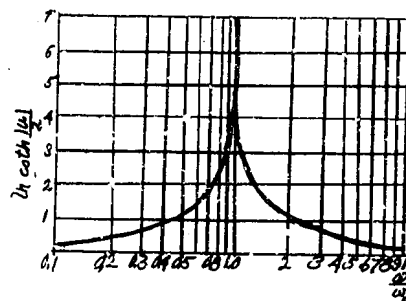


Figure 12. Graph of the Function
In coth $\frac{101}{2}$

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